電子情報通信特論
Advanced Course of Electronics, Information and Communications

Digital Modulation

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自然科学研究科　電気情報工学専攻

NiiGATA UNiVERSiTY

Outline

- Signal Space Analysis
- Modulation Schemes
  - Amplitude/Phase Modulation
  - Frequency Modulation
- Pulse Shaping

Digital Modulation

- Benefits
  - Higher spectral efficiency
  - Powerful error correction
  - Resistance to channel impairments
- Linear type: Amplitude and phase modulation
  - Better spectral efficiency
  - Not robust against channel impairments
  - Needs linear amplifiers (expensive and low power efficiency)
- Nonlinear type: Frequency modulation
  - Worse spectral efficiency
  - Robust against channel impairments
  - Nonlinear amplifier is enough (cheap and high power efficiency)

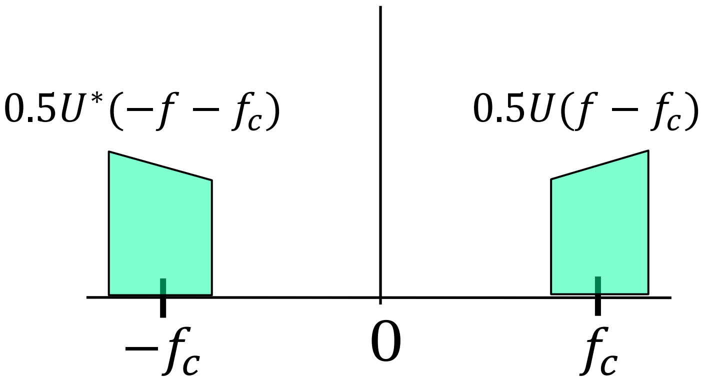
Modulated Signal Representation

- Bandpass signal (passband; real signal)

$$s(t) = A(t)\cos(2\pi f_c t + \theta(t)) = \mathrm{Re}\left[e^{\mathrm{j}2\pi f_c t} A(t) e^{\mathrm{j}\theta(t)}\right]$$

- Lowpass signal (baseband; complex signal)

$$u(t) = A(t)e^{\mathrm{j}\theta(t)} = A(t)(\cos\theta(t) + \mathrm{j}\sin\theta(t))$$



$$\begin{aligned} s(t) &= \mathrm{Re}\left[u(t)e^{\mathrm{j}2\pi f_c t}\right] \\ &= u_{\mathrm{I}}(t)\cos 2\pi f_c t - u_Q(t)\sin 2\pi f_c t \\ &= \frac{1}{2}\left(u(t)e^{\mathrm{j}2\pi f_c t} + u^*(t)e^{-\mathrm{j}2\pi f_c t}\right) \end{aligned}$$

$$\mathcal{F}\{s(t)\} = \frac{1}{2}\left(U(f - f_c) + U^*(-f - f_c)\right)$$

Quadrature Modulation

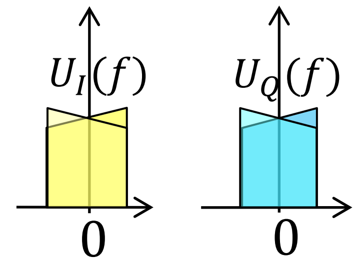
- Lowpass signal (baseband; complex signal)

$$\boxed{u(t) = u_{\mathrm{I}}(t) + \mathrm{j}u_Q(t)}$$

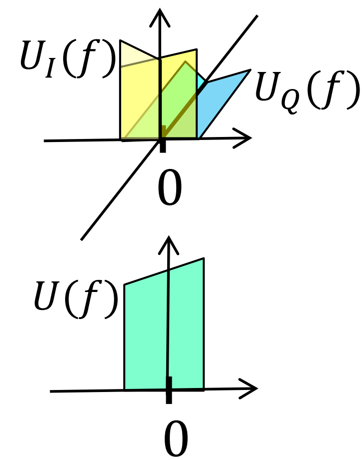
$$\mathcal{F}$$

$$U(f) = U_{\mathrm{I}}(f) + \mathrm{j}U_Q(f)$$
$$U^*(-f) = U_{\mathrm{I}}(f) - \mathrm{j}U_Q(f)$$

$$U_I(f) = 0.5[U(f) + U^*(-f)]$$
$$U_Q(f) = 0.5/\mathrm{j}[U(f) - U^*(-f)]$$

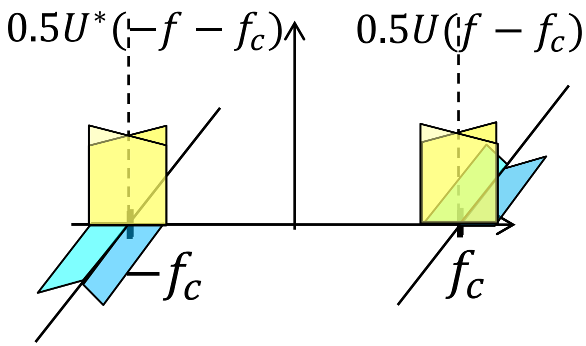


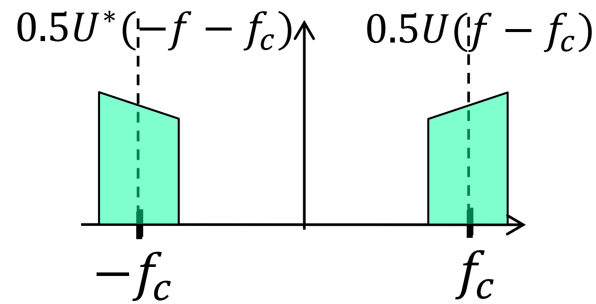
Conjugate symmetry



- Q-Mod and bandpass signal generation

$$\mathcal{F}\{\mathrm{Re}[u(t)e^{\mathrm{j}2\pi f_c t}]\} = 0.5(U(f - f_c) + U^*(-f - f_c))$$





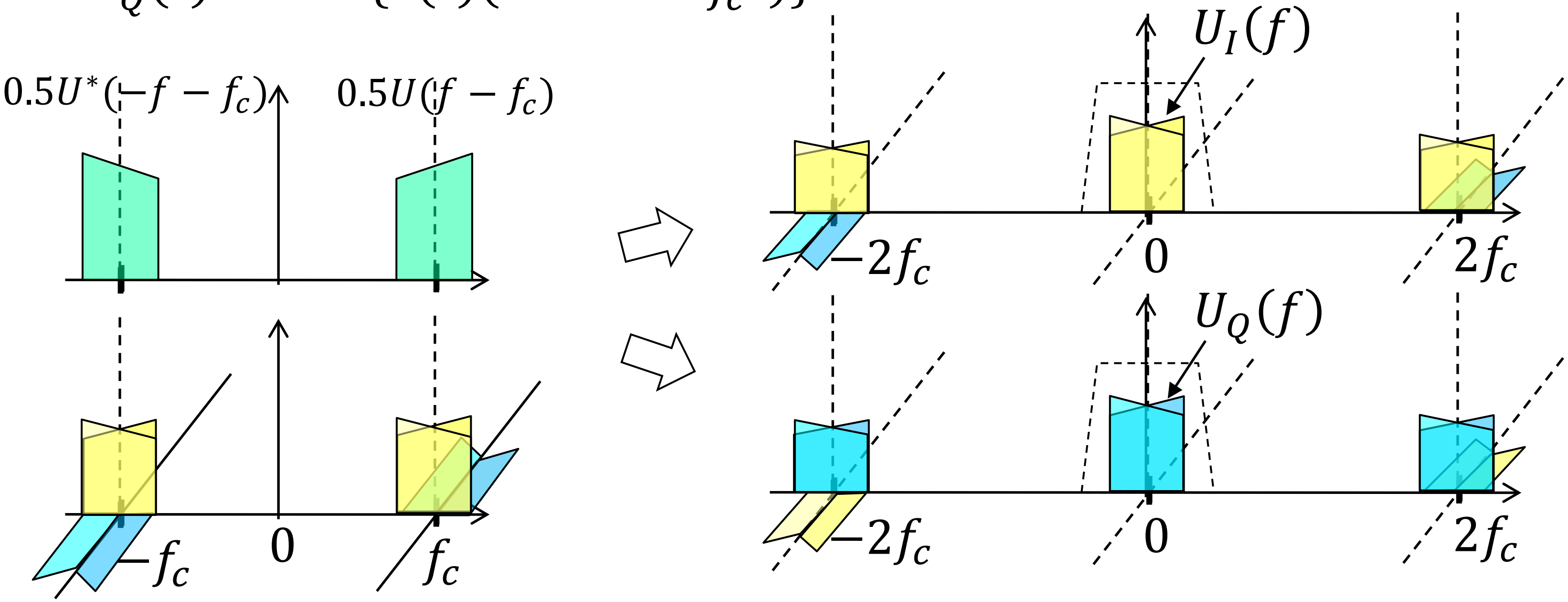
Quadrature Demodulation

- Bandpass signal

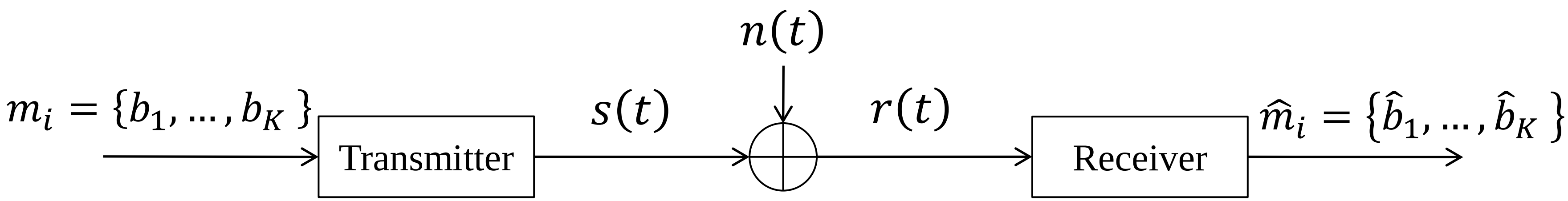
$$r(t) = \mathrm{Re}\left[u(t)e^{\mathrm{j}2\pi f_c t}\right] = u_\mathrm{I}(t)\cos 2\pi f_c t - u_Q(t)\sin 2\pi f_c t$$
$$= \frac{1}{2}\left(u(t)e^{\mathrm{j}2\pi f_c t} + u^*(t)e^{-\mathrm{j}2\pi f_c t}\right)$$

- Q-demod and lowpass signal extraction

$$u_I(t) = LPF\{r(t)(2\cos 2\pi f_c t)\}$$
$$u_Q(t) = LPF\{r(t)(-2\sin 2\pi f_c t)\}$$



System Model (AWGN)

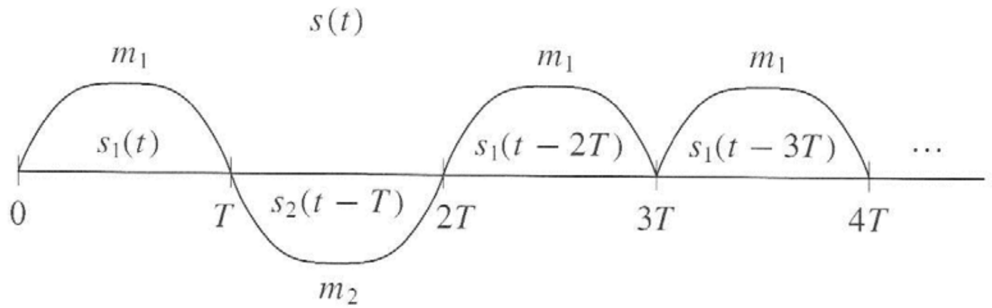


- Symbol period: T seconds
- Data per unit time: $K = \log_2 M$ [bits]
- Data rate: $R = \frac{K}{T}$ [bps]
- There are $M = 2^K$ possible sequences of K bits
  - A message $m_i = \{b_1, \ldots, b_K\} \in \mathcal{M}$ is comprised of a bit sequence of length K where $\mathcal{M} = \{m_1, \ldots, m_M\}$
  - The messages have probability p_i of being selected for transmission, where $\sum p_i = 1$

System Model (AWGN)

- Suppose message m_i is to be transmitted over the channel during time interval $[0, T)$
- Each message $m_i \in \mathcal{M}$ is mapped to a unique signal $s_i \in S$ where $S = \{s_1(t), \dots, s_M(t)\}$
- Energy

$$E_{s_i} = \int_0^T s_i^2(t)dt, \qquad i = 1, \dots, M$$



- Detection

$$P_e = \sum_{i=1}^{M} p(\hat{m} \neq m_i | m_i \text{ sent}) p(m_i \text{ sent})$$

Geometrical Representation

- Any set of M real signals defined on $[0, T)$ with finite energy can be represented as a linear combination of $N \leq M$ real orthonormal basis functions $\{\phi_1(t), \dots, \phi_N(t)\}$ as

$$s_i(t) = \sum_{j=1}^{N} s_{ij} \phi_j(t)$$

Gram-Schmidt
Orthogonalization procedure

where the coefficient, $s_{ij} = \int_0^T s_i(t) \phi_j(t)\, dt$

- Basis functions span the set S and

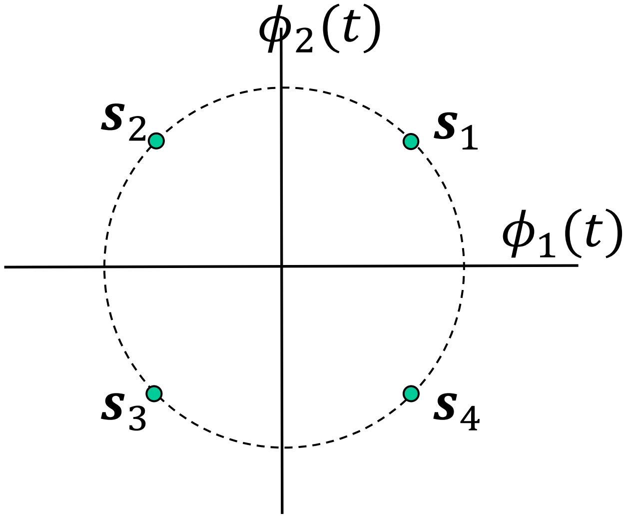
$$\int_0^T \phi_i(t) \phi_j(t)\, dt = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Signal Space

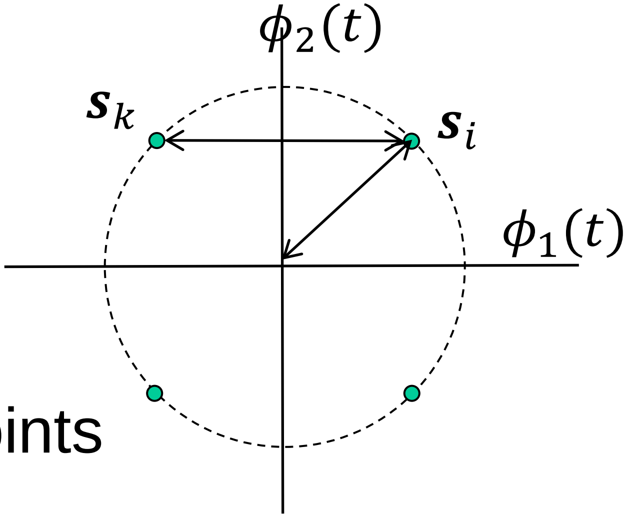
- Minimum number $N \backsimeq 2BT$ ($s_i(t)$ occupies $2BT$ orthogonal dimension)
- Linear passband modulation

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos 2\pi f_c t, \quad \phi_2(t) = \sqrt{\frac{2}{T}} \sin 2\pi f_c t$$

- Signal constellation
  - $\boldsymbol{s}_i = [s_{i1}, \ldots, s_{iN}]^T \in \mathbb{R}^N$
  - $s_i(t)$ is represented by $\boldsymbol{s}_i$ (signal space representation)
  - MPSK and MQAM → 2-dim (in-phase and quadrature)



Signal Space Representation



- Length of the vector

$$\|\boldsymbol{s}_i\| \triangleq \sqrt{\sum_{j=1}^{N} s_{ij}^2}$$

- Distance between two constellation points

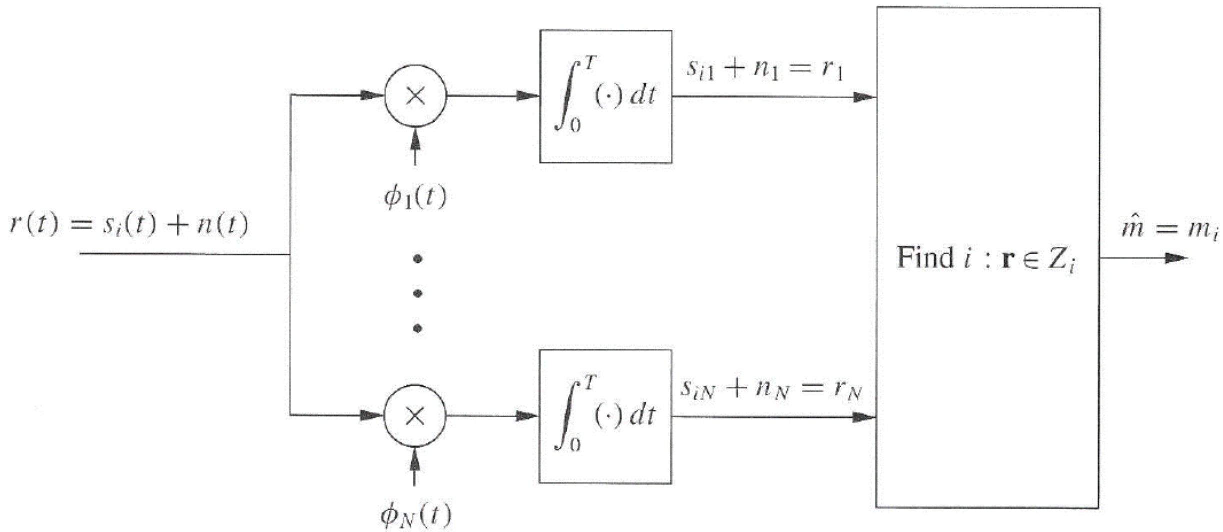
$$\|\boldsymbol{s}_i - \boldsymbol{s}_k\| = \sqrt{\sum_{j=1}^{N} \left(s_{ij} - s_{kj}\right)^2} = \sqrt{\int_0^T (s_i(t) - s_k(t))^2 dt}$$

- Inner product

$$\langle \boldsymbol{s}_i, \boldsymbol{s}_k \rangle = \boldsymbol{s}_i^T \boldsymbol{s}_k = \int_0^T s_i(t) s_k(t) dt = \langle s_i(t), s_k(t) \rangle$$

Receiver Structure

- Receiver structure for signal detection in AWGN



$$r(t) = \sum_{j=1}^{N} (s_{ij} + n_j)\phi_j(t) + n_r(t) = \sum_{j=1}^{N} r_j\phi_j(t) + n_r(t)$$

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \quad n_j = \int_0^T n(t)\phi_j(t)dt, \quad n_r(t) = n(t) - \sum_{j=1}^{N} n_j\phi_j(t)$$

$\boldsymbol{r} = [r_1, \dots, r_N]$ is *sufficient statistics* for $r(t)$ in the optimum detection

Decision Regions and MLD

- The optimum receiver minimizes error prob. by selecting the detector output $\hat{m}$ → maximizes $1 - P_e = p(\hat{m}\ \text{sent}|\boldsymbol{r})$
- Decision region

$$Z_i = \{\boldsymbol{r}: p(\boldsymbol{s}_i\ \text{sent}|\boldsymbol{r}) > p(\boldsymbol{s}_j\ \text{sent}|\boldsymbol{r})\ \forall j \neq i\}$$

- Bayes' rule

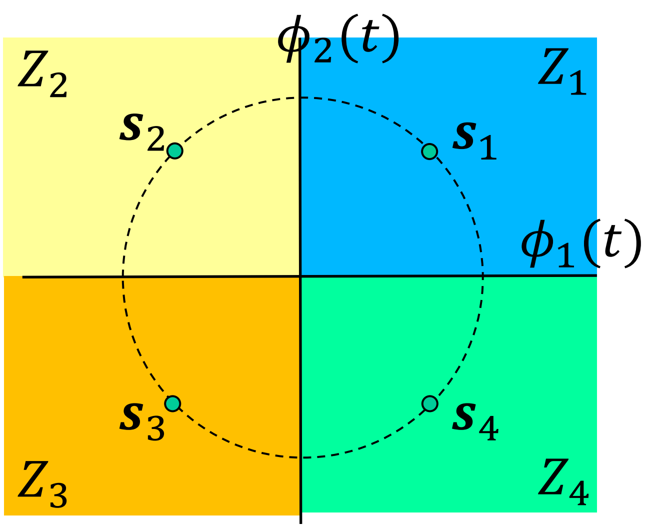
$$p(\boldsymbol{s}_i|\boldsymbol{r}) = \frac{p(\boldsymbol{r}|\boldsymbol{s}_i)p(\boldsymbol{s}_i)}{p(\boldsymbol{r})}$$

- Maximum likelihood detection

$$\arg\max_{\boldsymbol{s}_i} p(\boldsymbol{s}_i|\boldsymbol{r}) \leftrightarrow \boxed{\arg\max_{\boldsymbol{s}_i} p(\boldsymbol{r}|\boldsymbol{s}_i)}$$

$$L(\boldsymbol{s}_i) = p(\boldsymbol{r}|\boldsymbol{s}_i) \quad \text{likelihood function}$$

Decision region



Maximum Likelihood Receiver

- Conditioned distribution of $\boldsymbol{r}$

$$p(\boldsymbol{r}|\boldsymbol{s}_i) = \prod_{j=1}^{N} p(r_j|m_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}\left(r_i - s_{ij}\right)^2\right]$$

where r_i is a Gauss-distributed RV that is independent of r_k $(k \neq j)$ with mean s_{ij} and variance $\frac{N_0}{2}$

- Log likelihood function

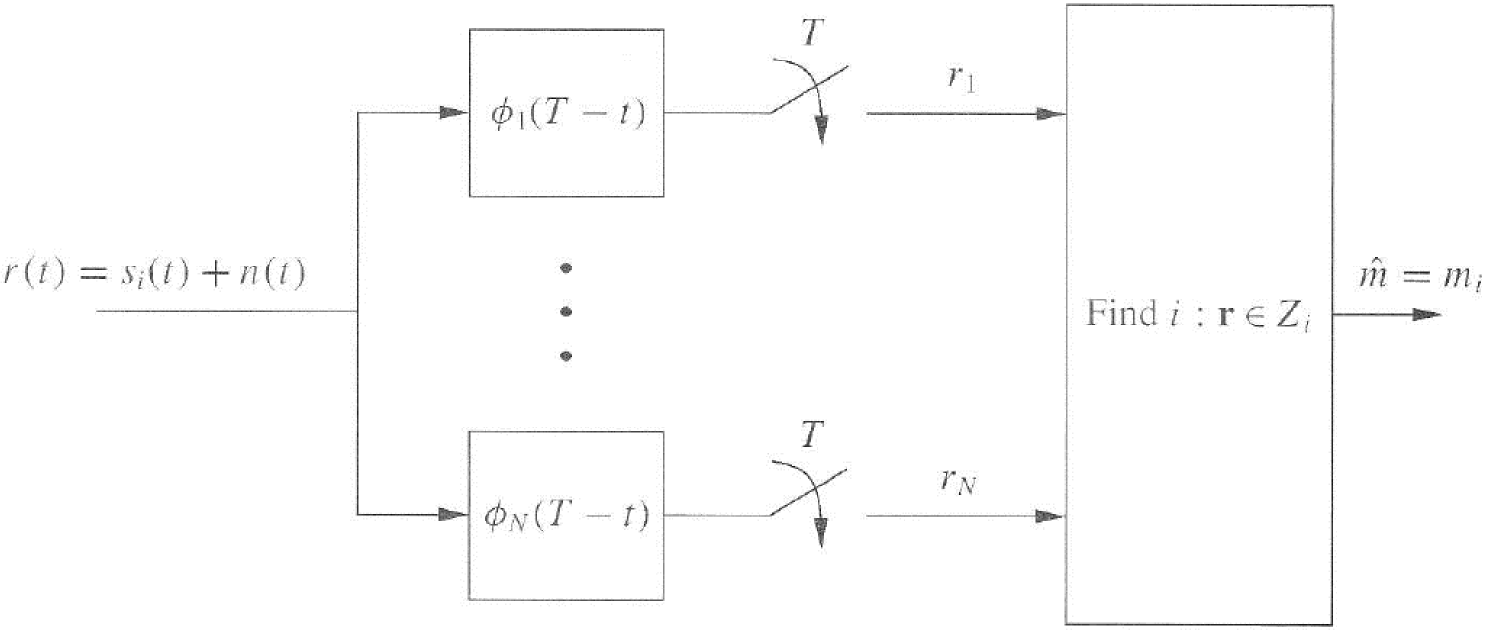
$$l(\boldsymbol{s}_i) = -\frac{1}{N_0}\sum_{j=1}^{N}\left(r_i - s_{ij}\right)^2 = -\frac{1}{N_0}\|\boldsymbol{r} - \boldsymbol{s}_i\|^2$$

- ML receiver

$$\arg\min_{\boldsymbol{s}_i}\|\boldsymbol{r} - \boldsymbol{s}_i\|^2$$

Matched Filter Receiver

- Matched filter receiver structure



$$y_j(t) = \int_0^t r(\tau)h_j(t-\tau)d\tau = \int_0^t r(\tau)\phi_j(T-t+\tau)d\tau$$

$$r_j = y_j(T) = \int_0^T r(\tau)\phi_j(\tau)d\tau$$ equivalent to correlation detector

Error Probability

- For equally likely messages, the error probability

$$P_e = \sum_{i=1}^{M} p(\boldsymbol{r} \notin Z_i | m_i \text{ sent}) p(m_i \text{ sent})$$

$$= 1 - \frac{1}{M} \sum_{i=1}^{M} p(\boldsymbol{r} \in Z_i | m_i \text{ sent}) \ = 1 - \frac{1}{M} \sum_{i=1}^{M} \int_{Z_i} p(\boldsymbol{r}|m_i) d\boldsymbol{r}$$

$$= 1 - \frac{1}{M} \sum_{i=1}^{M} \int_{Z_i} p(\boldsymbol{r} = \boldsymbol{s}_i + \boldsymbol{n} | \boldsymbol{s}_i) d\boldsymbol{n} \ = 1 - \frac{1}{M} \sum_{i=1}^{M} \int_{Z_i - \boldsymbol{s}_i} p(\boldsymbol{n}) d\boldsymbol{n}$$

- Error probability conditioned that m_i is sent

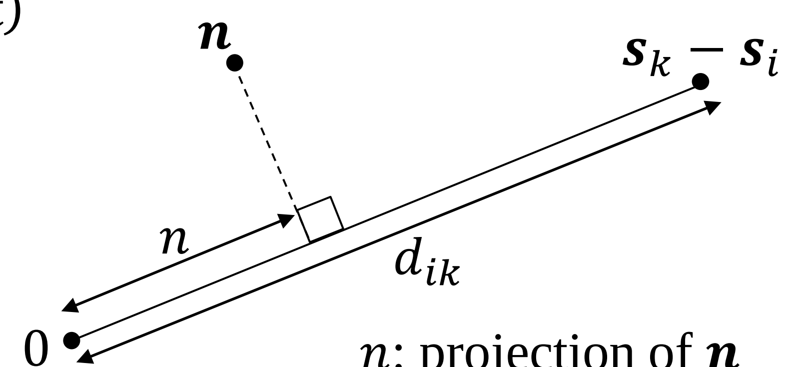
$$P_e(m_i \text{ sent}) = p\left(\bigcup_{\substack{k=1 \\ k \neq i}}^{M} A_{ik} \right) \leq \sum_{\substack{k=1 \\ k \neq i}}^{M} p(A_{ik})$$

where A_{ik} denotes the event that $\|\boldsymbol{r} - \boldsymbol{s}_k\| < \|\boldsymbol{r} - \boldsymbol{s}_i\|$

Error Probability

- Prob. event A_{ik}

$$p(A_{ik}) = p(\|\boldsymbol{s}_k - \boldsymbol{r}\| < \|\boldsymbol{s}_i - \boldsymbol{r}\| \,|\, \boldsymbol{s}_i \text{ sent})$$
$$= p(\|\boldsymbol{s}_k - \boldsymbol{s}_i - \boldsymbol{n}\| < \|\boldsymbol{n}\|)$$



n: projection of $\boldsymbol{n}$ onto the line

$$p(A_{ik}) = p\left(n > \frac{d_{ik}}{2}\right)$$
$$= \int_{\frac{d_{ik}}{2}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{v^2}{N_0}\right] dv = Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

- Prob. error

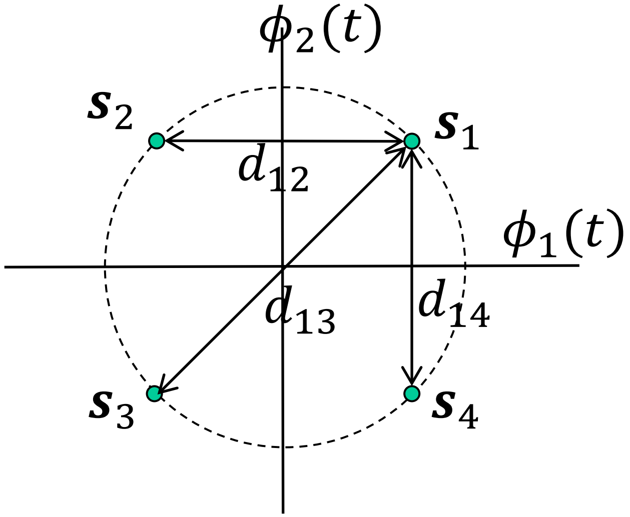
$$P_e(m_i \text{ sent}) \leq \sum_{\substack{k=1\\k\neq i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

$$P_e = \sum_{i=1}^{M} P_e(m_i \text{ sent}) p(m_i \text{ sent}) \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{k=1\\k\neq i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

Nearest Neighbor Approximation

- Introducing the minimum distance of the constellation as $d_{\min} = \min_{i,k} d_{ik}$, we can simplify with the looser bound

$$P_e \leq (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$



- Nearest neighbor approximation

$$\boxed{P_e \approx M_{d_{\min}} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)}$$

$M_{d_{\min}}$: the number of neighbors at $d_{\min}$

  - For BPSK, $M = 2$, and $M_{d_{\min}} = 1$ → $P_b = Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$

- By mapping gray code, $P_b \approx \frac{P_e}{\log_2 M}$

Various Modulation Schemes

- Bandpass signal

$$s(t) = A(t)\cos(2\pi f_c t + \theta(t)) = \mathrm{Re}\left[u(t)e^{\mathrm{j}2\pi f_c t}\right]$$

- Lowpass signal

$$u(t) = s_I(t) + \mathrm{j}s_Q(\mathrm{t}) = \boxed{A(t)}e^{\mathrm{j}\boxed{\theta(t)}}$$

$$\frac{1}{2\pi}\frac{d\theta(t)}{dt}$$

Amplitude Modulation (MPAM)

Phase Modulation (MPSK)

Frequency Modulation (FSK, MSK)

Amplitude & Phase Modulation (MQAM)

Amplitude and Phase Modulation

- Bandpass signal

$$s(t) = s_I(t)\cos 2\pi f_c t - s_Q(t)\sin 2\pi f_c t$$

- Signal space representation

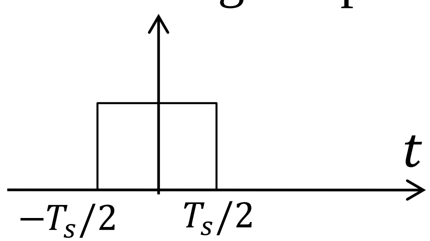
$$s(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t) \qquad i = 1, \dots, M$$

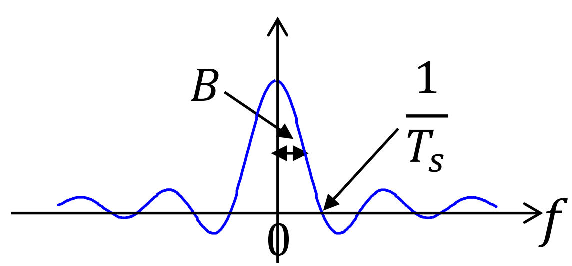
Base functions
$$\phi_1(t) = g(t)\cos(2\pi f_c t + \varphi_0)$$
$$\phi_2(t) = -g(t)\sin(2\pi f_c t + \varphi_0)$$

Baseband signals
$$s_I(t) = s_{i1} g(t)$$
$$s_Q(t) = s_{i2} g(t)$$

- Pulse shape $g(t)$ is a signal with bandwidth of B

Rectangular pulse



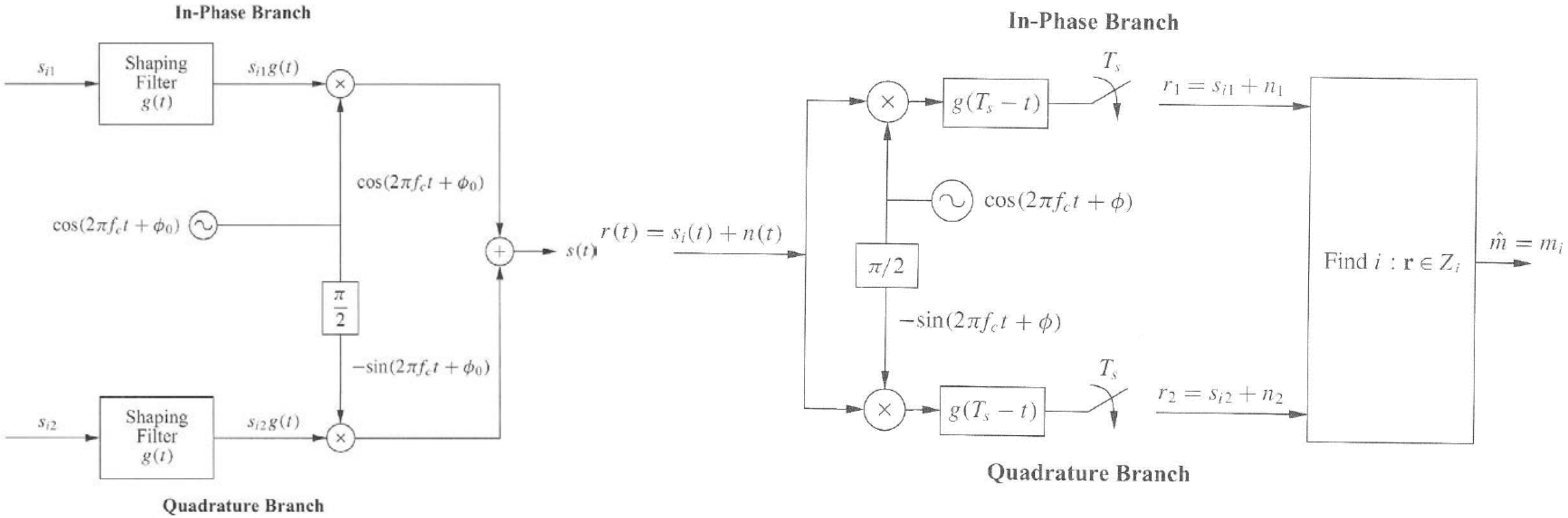


$$T_s = \frac{1}{2B}$$

Nyquist rate

Amplitude and Phase Modulation

- Amplitude and phase modulation / demodulation



- Carrier phase recovery: to match carrier phases (ϕ_0 and ϕ)

  → *Coherent detection*

- Synchronization or timing recovery: to find the start of the symbol period

Pulse Amplitude Modulation (MPAM)

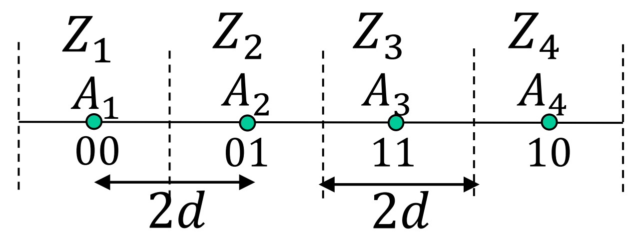
- Modulated signal

$$s_i(t) = \mathrm{Re}\left[A_i g(t) e^{j2\pi f_c t}\right] = A_i g(t)\cos 2\pi f_c t \qquad 0 \le t \le T_s \gg 1/f_c$$

$$A_i = (2i - 1 - M)d, \qquad i = 1, \dots, M$$

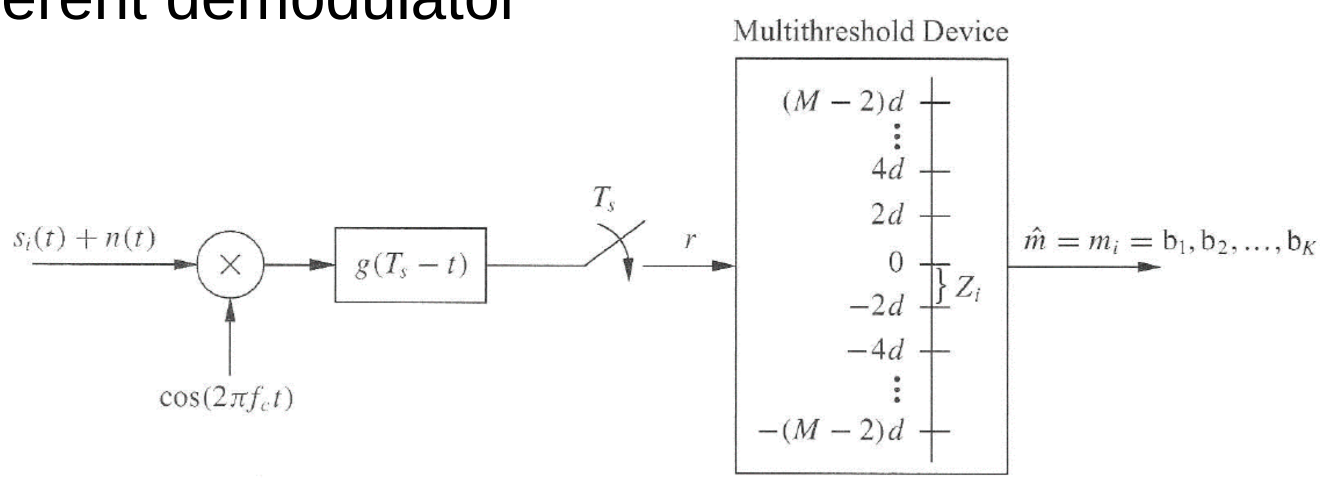
$$d_{\min} = 2d$$

$$E_{s_i} = \int_0^{T_s} s_i^2(t)dt = A_i^2 \qquad E_s = \frac{1}{M}\sum_{i=1}^{M} A_i^2$$



<4PAM($M = 4, K = 2$)>

- Coherent demodulator



Phase Shift Keying (MPSK)

- Modulated signal over one symbol time T_s

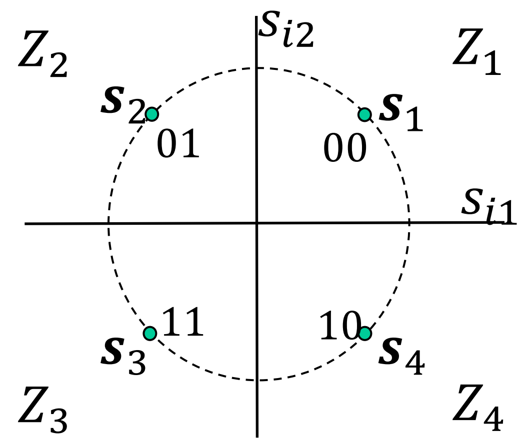
$$0 \le t \le T_s$$

$$s_i(t) = \mathrm{Re}\left[A e^{\mathrm{j}\left(2\pi\frac{(i-1)}{M}+\frac{\pi}{4}\right)} g(t) e^{\mathrm{j}2\pi f_c t}\right] = A g(t)\cos\left(2\pi f_c t + 2\pi\frac{(i-1)}{M} + \frac{\pi}{4}\right)$$

$$s_{i1}(t) = A\cos\left(2\pi\frac{(i-1)}{M} + \frac{\pi}{4}\right),\ s_{i2}(t) = A\sin\left(2\pi\frac{(i-1)}{M} + \frac{\pi}{4}\right)$$

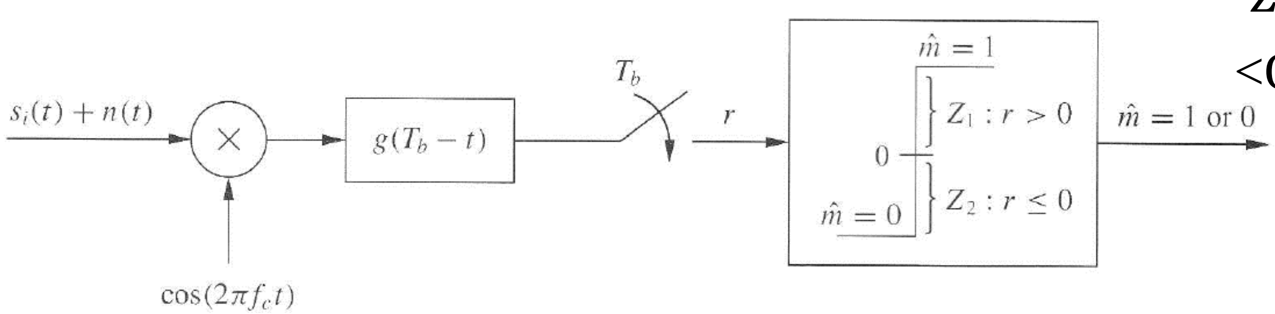
$$d_{\min} = 2A\sin\left(\frac{\pi}{M}\right)$$

$$E_{s_i} = \int_0^{T_s} s_i^2(t)dt = A^2 \ \ \text{(Equal energy)}$$



<QPSK($M = 4, K = 2$)>

- Coherent demodulator for BPSK



Quadrature Amplitude Modulation (MQAM)

- Modulated signal over one symbol time T_s

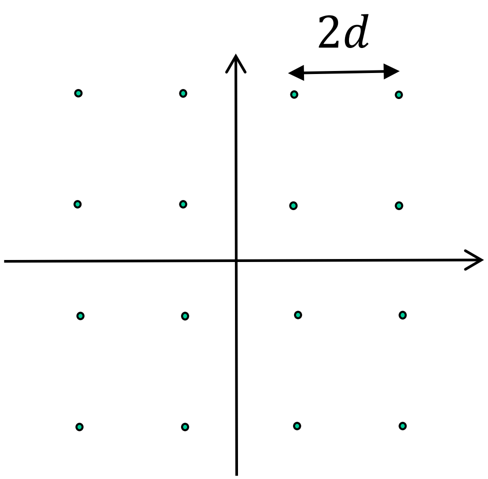
$$s_i(t) = \mathrm{Re}\left[A_i e^{\mathrm{j}\theta_i} g(t) e^{\mathrm{j}2\pi f_c t}\right] = A_i g(t)\cos(2\pi f_c t + \theta_i)$$

s_{i1}, and s_{i2} take values on $(2i-1-L)d$ for $i = 1, \dots, L$

→ 2-D extension of MPAM

$$d_{\min} = 2d$$

$$E_{s_i} = \int_0^{T_s} s_i^2(t)dt = A_i^2$$



<16QAM>

$(M = 16, K = 4,$
$l = 2, L = 4)$

l: bits per dimension
L^2 : size of constellation

- Average power of a square signal constellation with l bits/dim, P_l
  - To increase 1 bit/dim (2 bits/symbol) while maintaining the same $d_{\min}$, the power is needed 4 times as

$$P_{l+1} \approx 4P_l$$

Differential Modulation

- MPSK and MQAM needs carrier phase recovery
→ coherent detection (high complexity and large cost)
- Differential modulation uses the previous symbol as a phase reference for the current symbol (differential phase)
→ DMPSK or DPSK (DBPSK, DQPSK, etc.)
- DPSK Modulation

$$\boldsymbol{r}(k) = Ae^{\mathrm{j}(\theta(k)+\varphi_0)} + n(k)$$

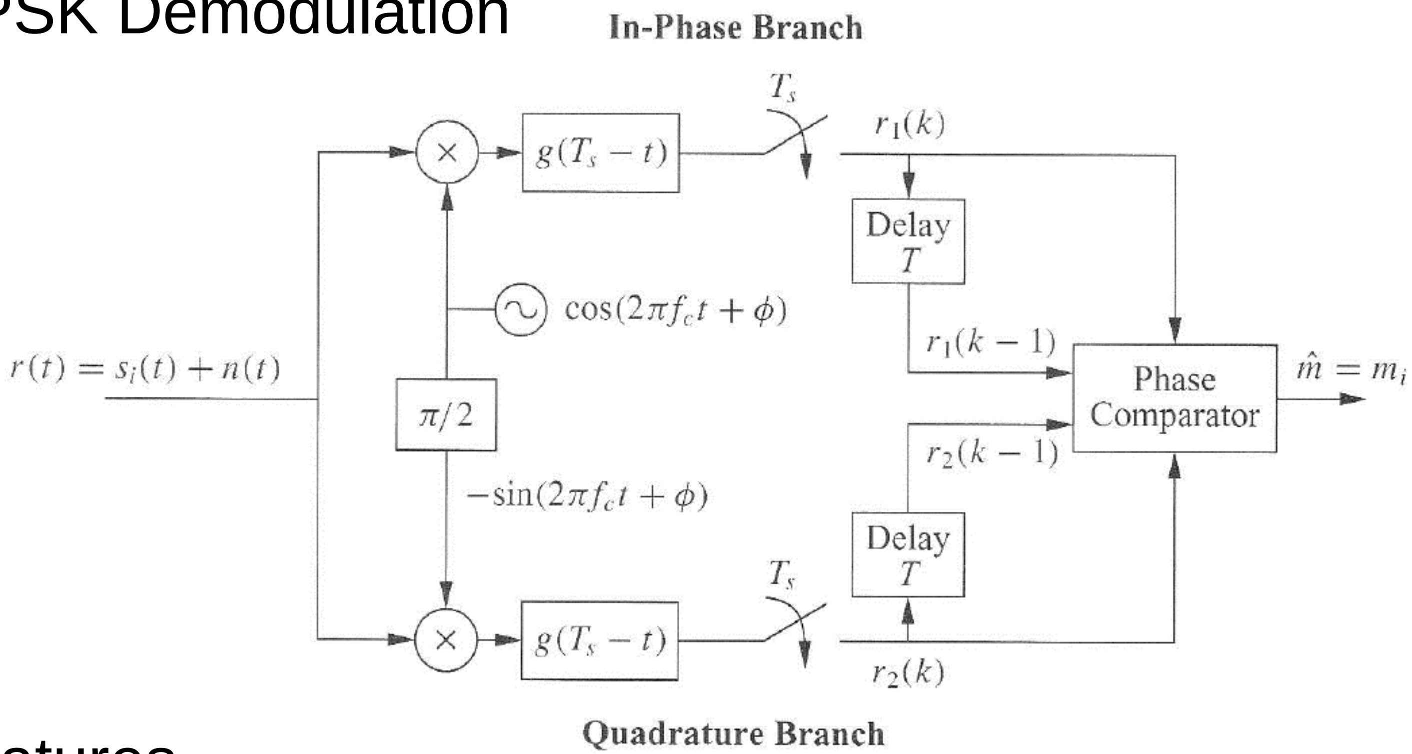
$n(k)$: complex WGN

$$\boldsymbol{r}(k-1) = Ae^{\mathrm{j}(\theta(k-1)+\varphi_0)} + n(k-1)$$

$$\boldsymbol{r}(k)\boldsymbol{r}^*(k-1) = \boxed{A^2 e^{\mathrm{j}(\theta(k)-\theta(k-1))}} + Ae^{\mathrm{j}(\theta(k)+\varphi_0)}n^*(k-1) + Ae^{-\mathrm{j}(\theta(k-1)+\varphi_0)}n(k) + n(k)n^*(k-1)$$

Differential Modulation

- DPSK Demodulation



- Features
  - Less sensitive to a random drift in the carrier phase
  - Degraded with non-zero Doppler frequency

Frequency Modulation

- Transmit signal over one symbol period

$$s_i(t) = A\cos(2\pi f_i t + \boxed{\varphi_i})$$

Phase associated with the i-th carrier

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t)$$

where $s_{ij} = A\delta(i-j)$

$\phi_j(t) = \cos\left(2\pi f_j t + \varphi_j\right)$ basis functions

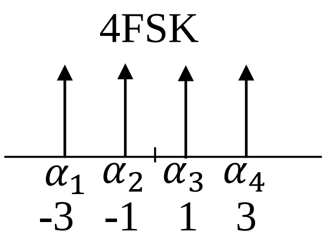
- Features
  - Constant envelope → nonlinear amplifiers can be used
  - Robust against amplitude distortion (due to fading or hardware)
  - Low spectral efficiency

Frequency Shift Keying (FSK)

- Modulated signal

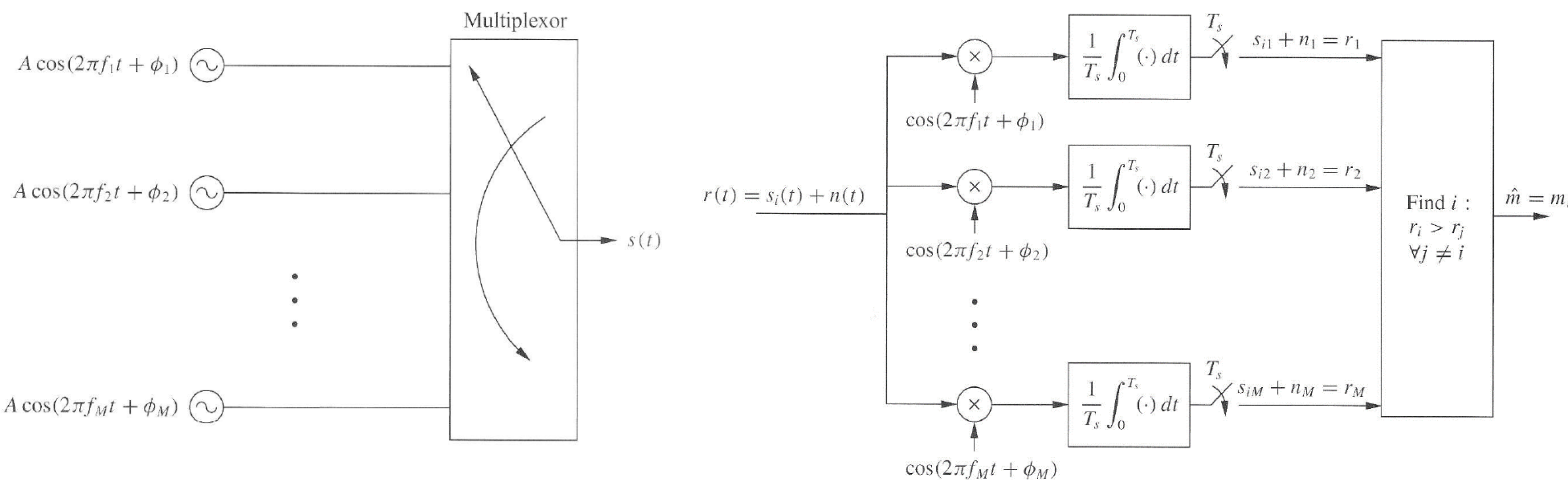
$$s_i(t) = A\cos(2\pi f_c t + 2\pi\alpha_i \Delta f_c t + \varphi_i), \qquad 0 \le t < T_s$$

where $\alpha_i = (2i - 1 - M)$ for $i = 1, \dots, M = 2^K$



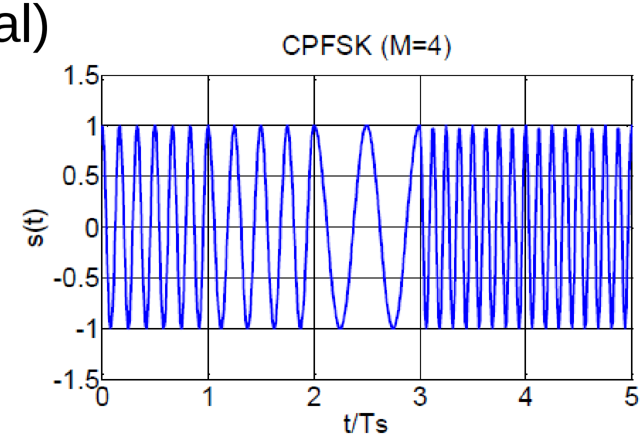
$$\phi_i(t) = \sqrt{2/T_s}\cos(2\pi f_c t + 2\pi\alpha_i \Delta f_c t + \varphi_i), \qquad i = 1, \dots, M$$

- Modulator/demodulator (coherent)

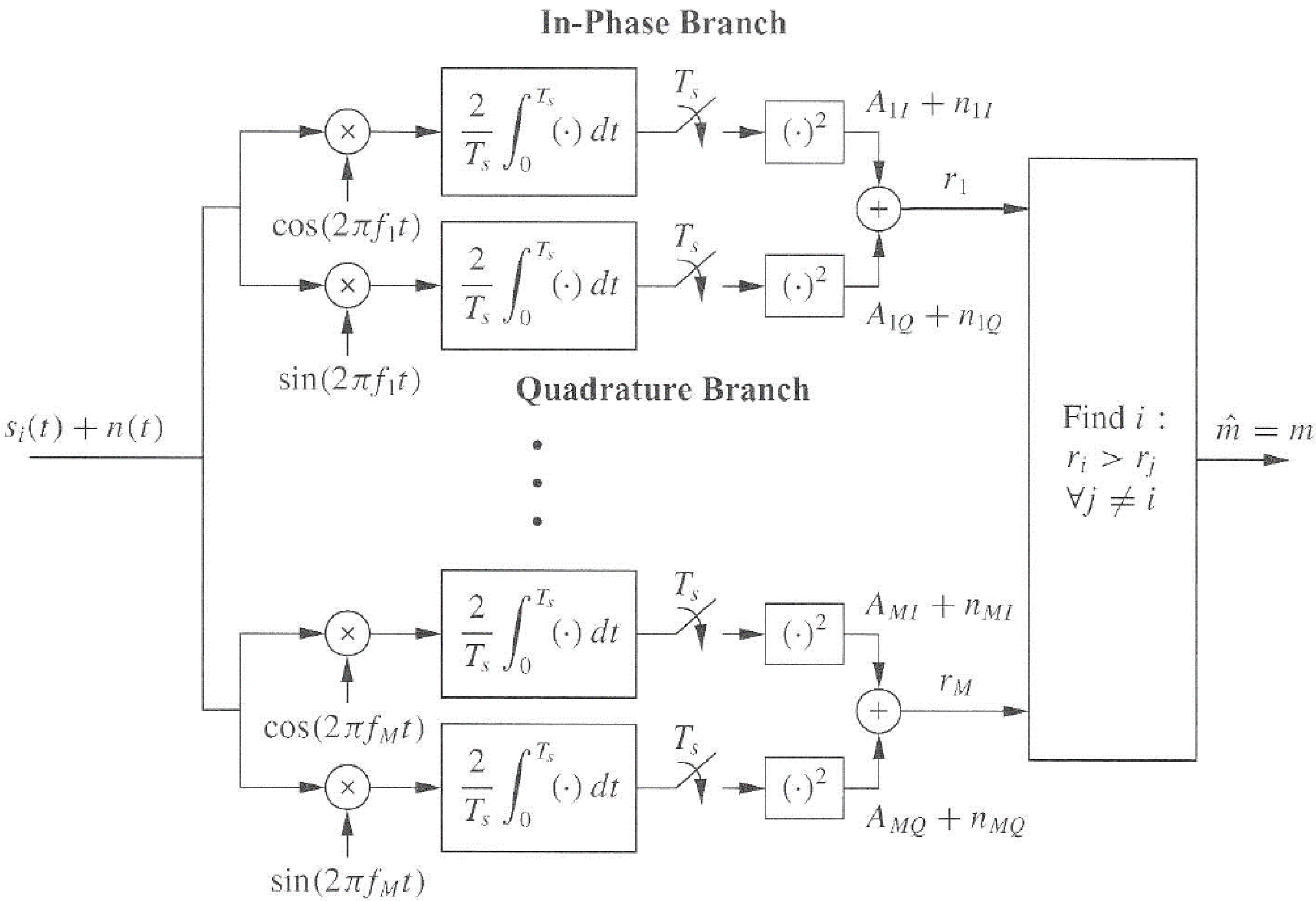


MSK and CPFSK

- MSK (minimum frequency shift keying)
  - Special case of binary FSK when $\varphi_i = \varphi_j$
  - $\Delta f = 2\Delta f_c = 0.5/T_s$ → minimum frequency separation for $\langle s_i(t), s_j(t) \rangle = 0$ for $i \neq j$
  - Minimum bandwidth
- CPFSK (Continuous-phase FSK)
  - Phase discontinuity → spectrum broadened
  - $s_i(t) = A\cos\left[2\pi f_c t + 2\pi\Delta f_c \int_{-\infty}^{t} u(\tau)d\tau\right] = A\cos\left(2\pi f_c t + \theta(t)\right)$ where $u(\tau) = \sum_k a_k\, g(t - kT)$ (MPAM signal)
  - $\theta(t)$ is continuous
  - Coherent detection in symbol-by-symbol or over a sequence of symbols is available

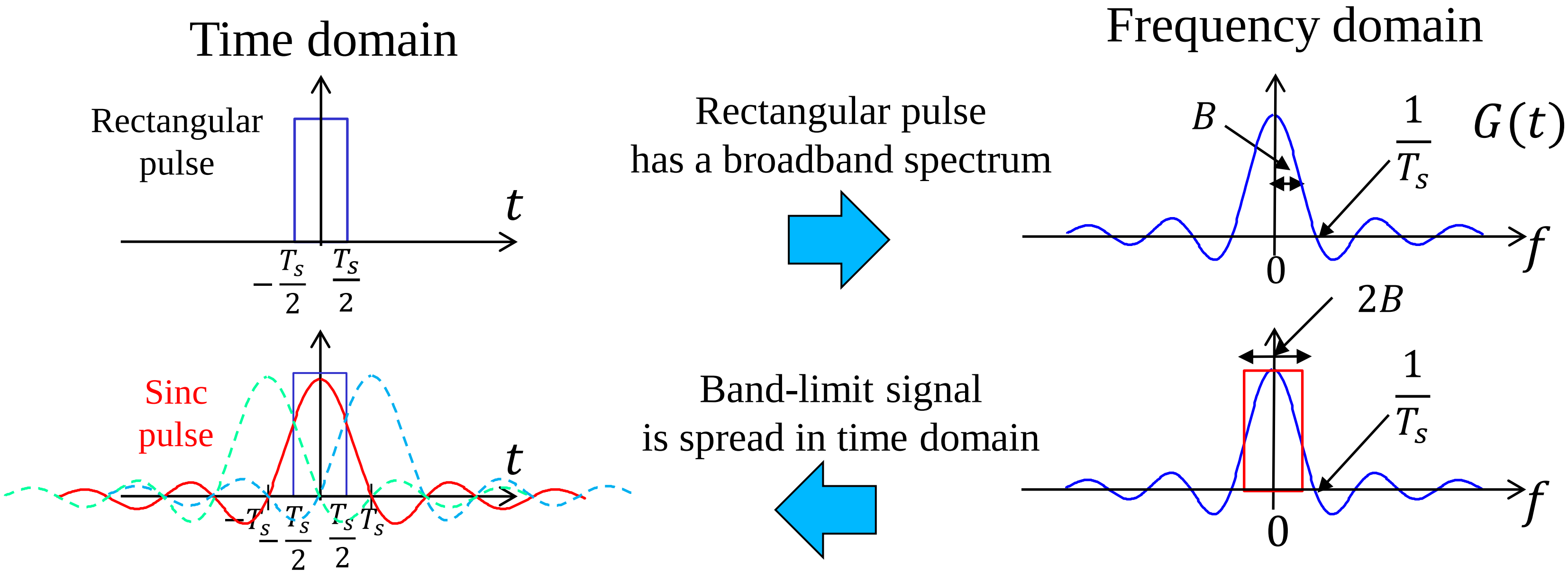


Noncoherent Detection for FSK



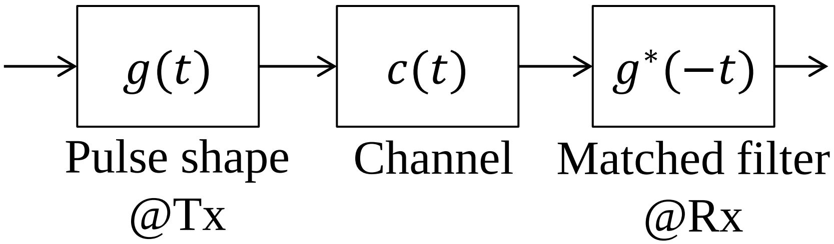
Pulse Shaping

- For amplitude and phase modulation, the signal bandwidth is determined by that of pulse shape $g(t)$
- Pulse shaping
  - Reduce the sidelobe energy (frequency domain property)
  - ISI (inter-symbol-interference) between pulses should be avoided (time domain property)



Optimum Filter Design

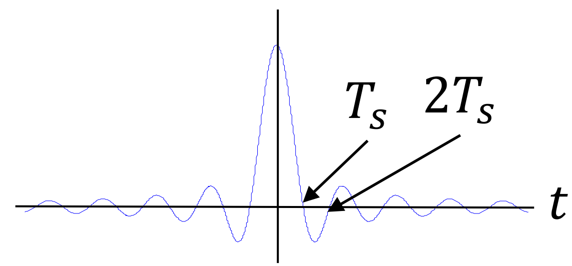
- Optimum TRx filter (pulse shaping) design → matched filter



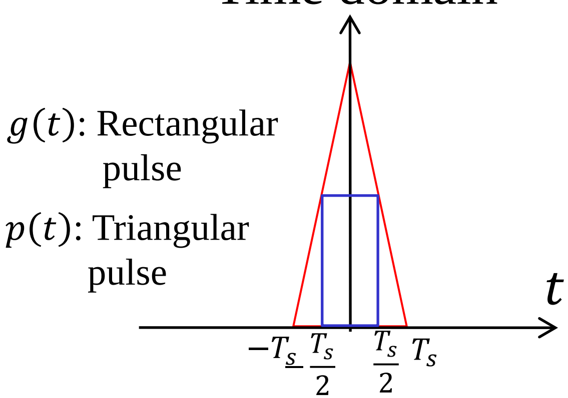
$$p(t) = g(t) * c(t) * g^*(-t)$$
$$= g(t) * g^*(-t)$$
when $c(t) = \delta(t)$ (AWGN)

Nyquist Criterion

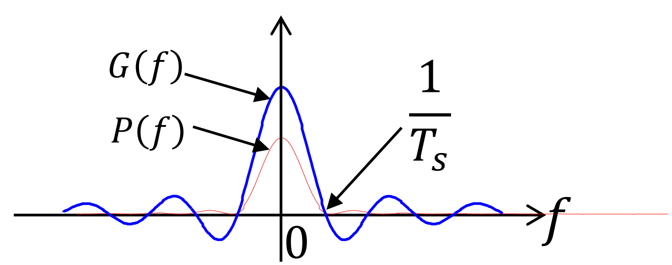
$$p(kT_s) = \begin{cases} p_0 = p(0) & k = 0 \\ 0 & k \neq 0 \end{cases}$$



Time domain



Frequency domain

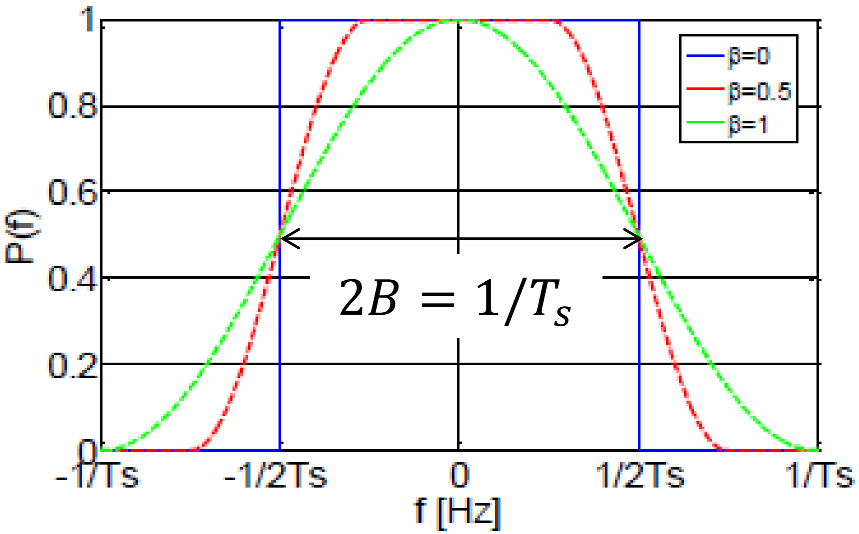


Raised Cosine Pulse

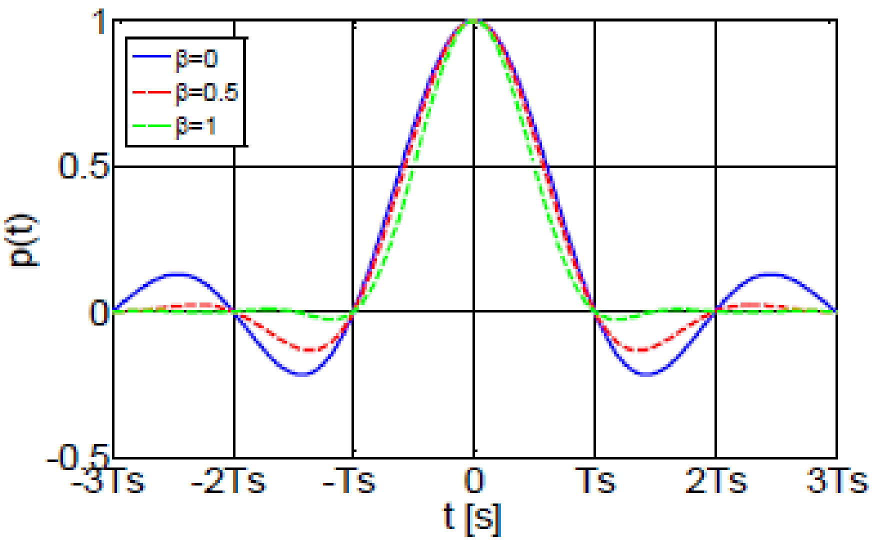
- Most commonly used
- Designed in frequency domain introducing roll-off β

$$P(f) = \begin{cases} T_s & 0 \le |f| \le \frac{1-\beta}{2T_s} \\ \frac{T_s}{2}\left[1 - \sin\frac{\pi T_s}{\beta}\left(f - \frac{1}{2T_s}\right)\right] & \frac{1-\beta}{2T_s} \le |f| \le \frac{1+\beta}{2T_s} \end{cases}$$

$$p(t) = \frac{\sin \pi t/T_s}{\pi t/T_s} \frac{\cos \beta\pi t/T_s}{1 - 4\beta^2 t^2/T_s^2}$$



frequency domain



Time domain

Outline

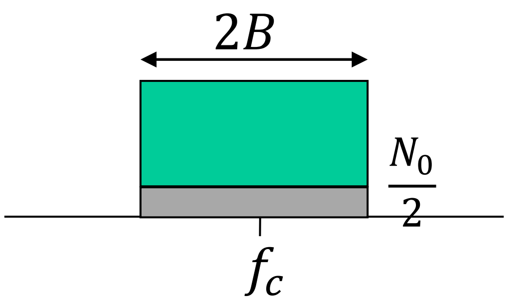
- Performance Criteria
  - Probability of error
  - Outage probability

- Error sources
  - Noise
  - Flat fading
  - Frequency selective fading → ISI (Inter symbol interference)
  - Time selective fading (Doppler) → ACI (Adjacent channel interference)

Signal to Noise Power Ratio (SNR)

- Signal to noise power ratio (SNR)

$$\mathrm{SNR} = \frac{P_r}{N_0 B} = \frac{E_s}{N_0 B T_s}$$



$$\gamma_s = \frac{E_s}{N_0} \qquad \gamma_b = \frac{E_b}{N_0}$$

For general pulses, $T_s = k/B$

$k = 1$ for pulse shaping with $\beta = 1$
$k = 0.5$ for rectangular pulse

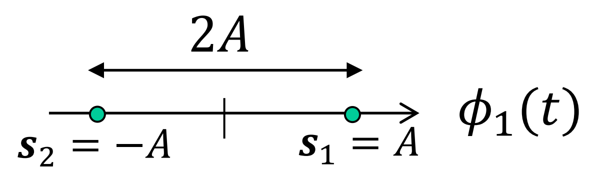
For M-ary signaling (MPAM and MPSK), $\gamma_b = \frac{\gamma_s}{\log_2 M}$

Error Probability in AWGN Channel

- Error probability for BPSK

$$P_s \approx M_{d_{\min}} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$P_b = P_s = Q\left(\sqrt{2\gamma_b}\right)$$

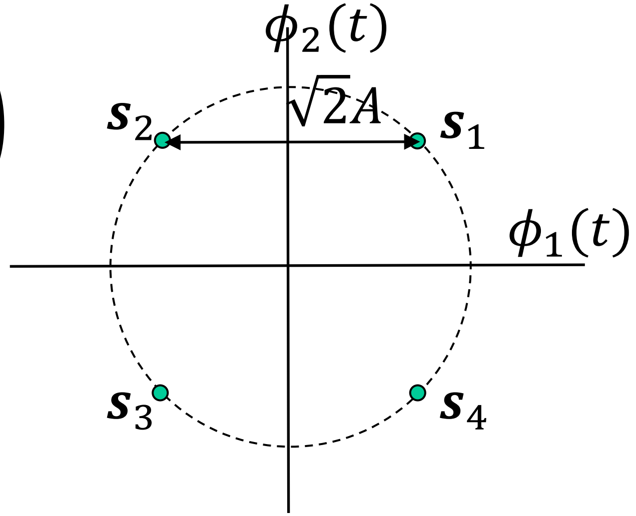


- Error probability for QPSK

$$P_s \approx M_{d_{\min}} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 2Q\left(\frac{\sqrt{2A^2}}{\sqrt{2N_0}}\right) = 2Q\left(\sqrt{\frac{A^2}{N_0}}\right)$$

$$= 2Q\left(\sqrt{\gamma_s}\right)$$

$P_b = Q\left(\sqrt{2\gamma_b}\right)$ for gray code



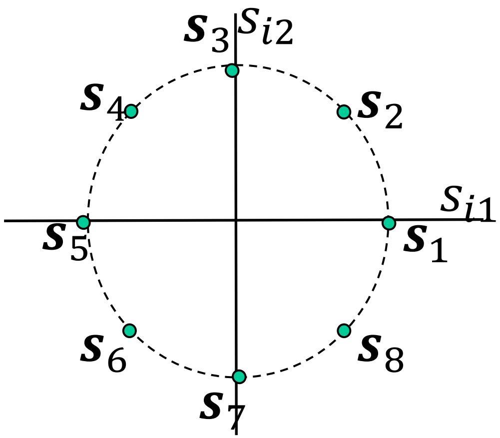
Error Probability in AWGN Channel

- Error probability for MPSK

$$P_s \approx M_{\mathrm{d_{min}}} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 2Q\left(\sqrt{2}\sin\frac{\pi}{M}\sqrt{\frac{A^2}{N_0}}\right) = 2Q\left(\sqrt{2\gamma_s}\sin\frac{\pi}{M}\right)$$

$$P_b = \frac{2}{\log_2 M} Q\left(2\sqrt{2\log_2 M\,\gamma_b}\sin\frac{\pi}{M}\right)$$

$$s_{i1}(t) = A\cos\left(2\pi\frac{(i-1)}{M}\right),$$
$$s_{i2}(t) = A\sin\left(2\pi\frac{(i-1)}{M}\right)$$

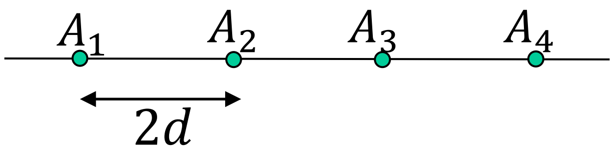


<8PSK($M = 8, K = 3$)>

Error Probability in AWGN Channel

- Error probability for MPAM

$$A_i = (2i - 1 - M)d, \quad i = 1, \dots, M$$



<4PAM($M = 4, K = 2$)>

$$P_s = \frac{1}{M}\sum_{i=1}^{M} P_s(\boldsymbol{s}_i)$$

$$P_s \approx \frac{M-2}{M} 2Q\left(\frac{2d}{\sqrt{2N_0}}\right) + \frac{2}{M} Q\left(\frac{2d}{\sqrt{2N_0}}\right)$$

$$= 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{2d^2}{N_0}}\right) = 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{1}{N_0}\frac{6\bar{E}_s}{M^2-1}}\right)$$

$$= 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6\bar{\gamma}_s}{M^2-1}}\right)$$

where $\bar{E}_s = \frac{1}{M}\sum_{i=1}^{M} A_i^2 = \frac{d^2}{M}\sum_{i=1}^{M}(2i - 1 - M)^2 = \frac{1}{3}(M^2 - 1)d^2$

$$P_b = \frac{2}{\log_2 M}\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6\bar{\gamma}_b \log_2 M}{M^2-1}}\right)$$

Differential Modulation

- DPSK

$$\boldsymbol{r}(k) = Ae^{\mathrm{j}(\theta(k)+\varphi_0)} + n(k)$$

$n(k)$: complex WGN

$$\boldsymbol{r}(k-1) = Ae^{\mathrm{j}(\theta(k-1)+\varphi_0)} + n(k-1)$$

$$\mathbf{z} = \boldsymbol{r}(k)\boldsymbol{r}^*(k-1) = \boxed{A^2e^{\mathrm{j}(\theta(k)-\theta(k-1))}}$$
$$+\underline{Ae^{\mathrm{j}(\theta(k)+\varphi_0)}n^*(k-1) + Ae^{-\mathrm{j}(\theta(k-1)+\varphi_0)}n(k)}$$
$$+\underline{n(k)n^*(k-1)}$$

Noise increases twice

very small negligible

- Error Probability

$$P_b = \frac{1}{2}e^{-\gamma_b} \quad \text{(DBPSK)}$$

$$P_b = \int_b^{\infty} x\exp\left[-\frac{(a^2+x^2)}{2}\right] I_0(ax)dx - \frac{1}{2}\int_b^{\infty} x\exp\left[-\frac{(a^2+b^2)}{2}\right] I_0(ab)dx \quad \text{(DQPSK)}$$

where $a \approx 0.765\sqrt{\gamma_b}$, $b \approx 1.85\sqrt{\gamma_b}$

Error Probability in AWGN (Summary)

- Coherent detection: perfect carrier phase recovery

Modulation	$P_s(\gamma_s)$	$P_b(\gamma_b)$
BFSK:		$P_b = Q\left(\sqrt{\gamma_b}\right)$
BPSK:		$P_b = Q\left(\sqrt{2\gamma_b}\right)$
QPSK,4QAM:	$P_s \approx 2\,Q\left(\sqrt{\gamma_s}\right)$	$P_b \approx Q\left(\sqrt{2\gamma_b}\right)$
MPAM:	$P_s \approx \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\overline{\gamma}_s}{M^2-1}}\right)$	$P_b \approx \frac{2(M-1)}{M\log_2 M} Q\left(\sqrt{\frac{6\overline{\gamma}_b \log_2 M}{(M^2-1)}}\right)$
MPSK:	$P_s \approx 2Q\left(\sqrt{2\gamma_s}\sin(\pi/M)\right)$	$P_b \approx \frac{2}{\log_2 M} Q\left(\sqrt{2\gamma_b \log_2 M}\sin(\pi/M)\right)$
Rectangular MQAM:	$P_s \approx \frac{4(\sqrt{M}-1)}{\sqrt{M}} Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)$	$P_b \approx \frac{4(\sqrt{M}-1)}{\sqrt{M}\log_2 M} Q\left(\sqrt{\frac{3\overline{\gamma}_b \log_2 M}{(M-1)}}\right)$
Nonrectangular MQAM:	$P_s \approx 4Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)$	$P_b \approx \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\overline{\gamma}_b \log_2 M}{(M-1)}}\right)$

Table 6.1: Approximate Symbol and Bit Error Probabilities for Coherent Modulations

$$P_s \approx \alpha_M Q\left(\sqrt{\beta_M \gamma_s}\right)$$

$$P_b \approx \frac{\alpha_M}{\log_2 M} Q\left(\sqrt{\beta_M \gamma_b \log_2 M}\right)$$

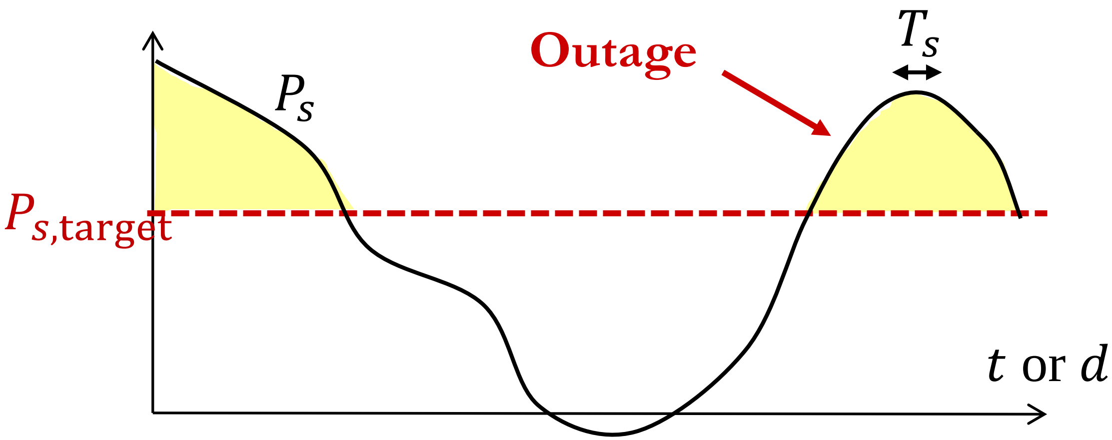
Fading Channel

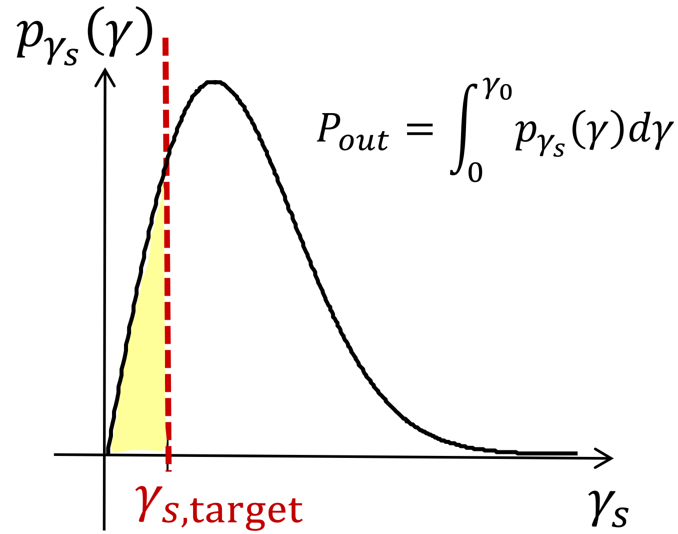
- Received signal power varies randomly over distance or time as a result of shadowing and multipath fading

  → γ_s is a random variable with $p_{\gamma_s}(\gamma)$, and so $P_s(\gamma_s)$ is also random

- Performance criteria
  - Outage probability $P_{\text{out}} = \Pr\left(P_s > P_{\text{target}}\right) = \Pr\left(\gamma_s < \gamma_{\text{target}}\right)$
  - Average error probability $\bar{P}_s = \int_0^\infty P_s(\gamma) p_{\gamma_s}(\gamma)\, d\gamma$
  - Combined average error probability and outage probability

Outage Probability

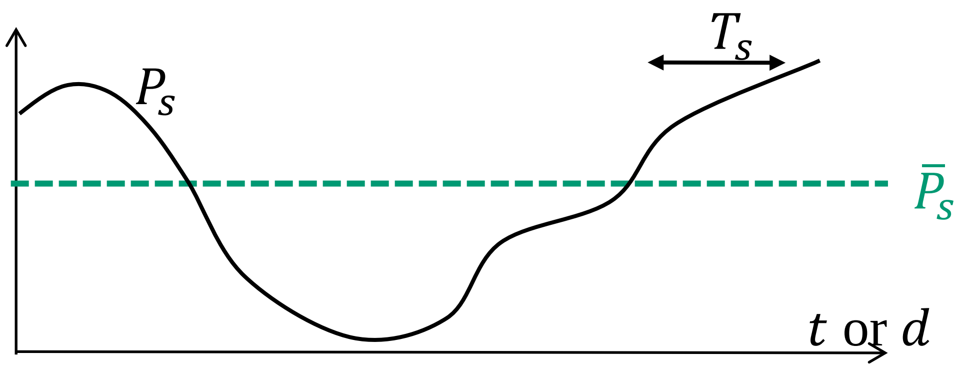




- Probability that P_s is above target
- Equivalently, probability γ_s below target
- Used when $T_c \gg T_s$

$$P_{out} = \Pr\left(P_s > P_{\text{target}}\right)$$
$$= \Pr\left(\gamma_s < \gamma_{\text{target}}\right)$$

Average P_s



$$\bar{P}_s = \int_0^{\infty} P_s(\gamma) p_{\gamma_s}(\gamma) d\gamma$$

- Expected value of random variable P_s
- Used when $T_c \approx T_s$
- Error probability much higher than in AWGN alone

Average P_s

- Rayleigh fading

$$p_r(r) = \frac{r}{\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right] \qquad \gamma = \frac{r^2 T_s}{2\sigma_n^2}$$

$$p_{\gamma_s}(\gamma) = \frac{\sigma_n^2}{\sigma^2 T_s}\exp\left[-\frac{\gamma\sigma_n^2}{\sigma^2 T_s}\right] = \frac{1}{\bar{\gamma}_s}\exp\left[-\frac{\gamma}{\bar{\gamma}_s}\right]$$

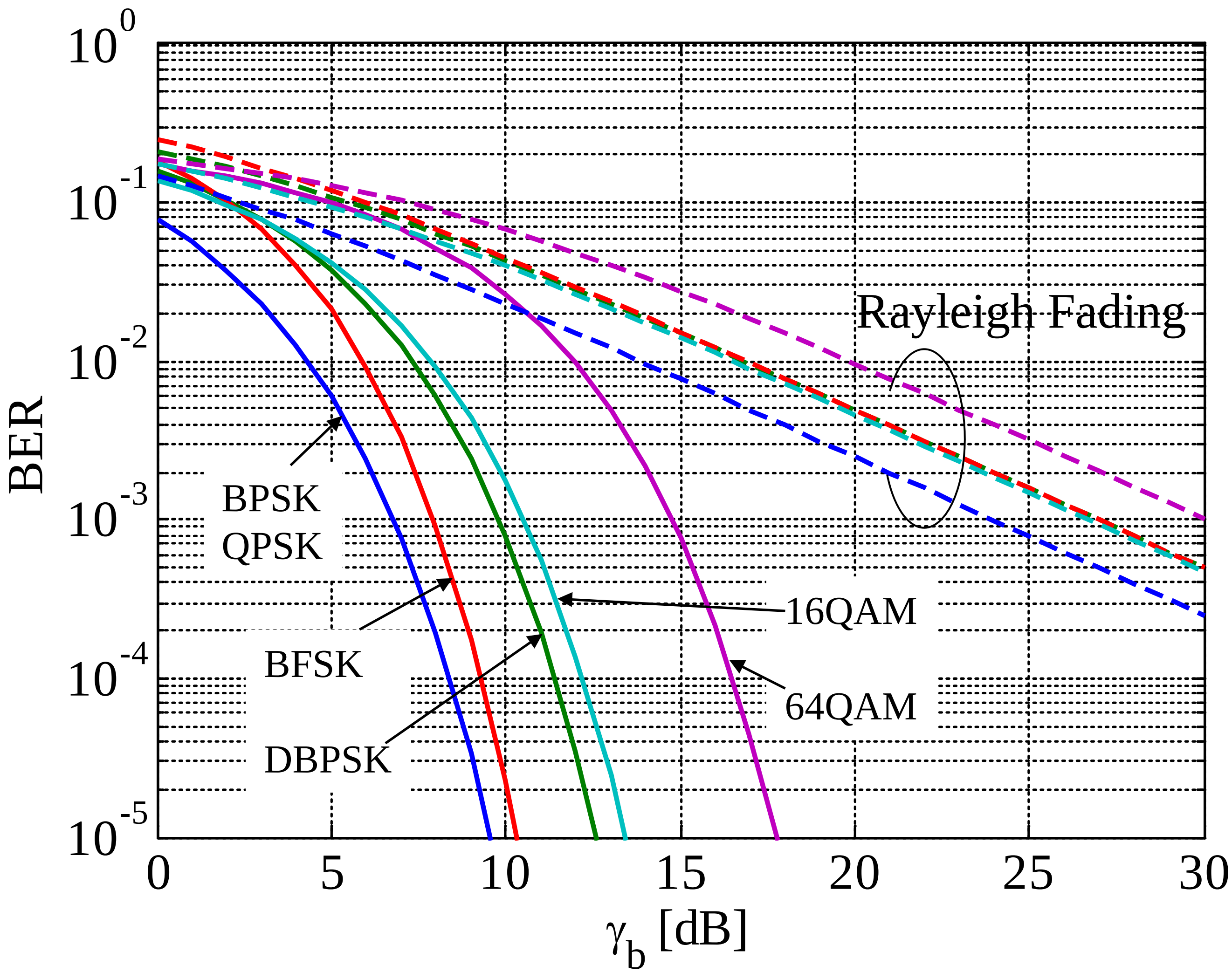
- Example average BER

$$\bar{P}_b = \int_0^\infty P_b(\gamma) p_{\gamma_b}(\gamma) d\gamma$$

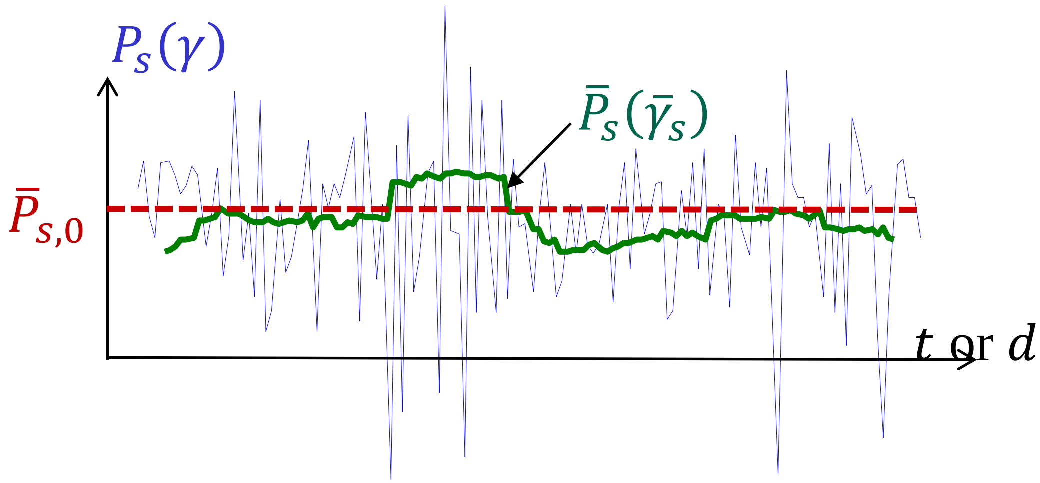
  - BPSK: $P_b(\gamma) = Q\left(\sqrt{2\gamma_b}\right)$, $\bar{P}_b = \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma}_b}{1+\bar{\gamma}_b}}\right] \approx \frac{1}{4\bar{\gamma}_b}$
  - BFSK: $P_b(\gamma) = Q\left(\sqrt{\gamma_b}\right)$, $\bar{P}_b = \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma}_b}{2+\bar{\gamma}_b}}\right] \approx \frac{1}{4\bar{\gamma}_b}$
  - DBPSK: $P_b(\gamma) = \frac{1}{2}e^{-\gamma_b}$, $\bar{P}_b = \frac{1}{2(1+\bar{\gamma}_b)} \approx \frac{1}{2\bar{\gamma}_b}$
  - General: $P_s(\gamma) \approx \alpha_M Q\left(\sqrt{\beta_M \gamma_s}\right)$, $\bar{P}_s = \frac{\alpha_M}{2}\left[1 - \sqrt{\frac{0.5\beta_M\bar{\gamma}_b}{1+0.5\beta_M\bar{\gamma}_b}}\right] \approx \frac{\alpha_M}{2\beta_M\bar{\gamma}_b}$

Average P_s

- AWGN vs Rayleigh fading



Combined Outage and Average P_s



$$\bar{P}_s(\bar{\gamma}_s) = \int_0^\infty P_s(\gamma_s) p(\gamma_s | \bar{\gamma}_s) d\gamma_s$$

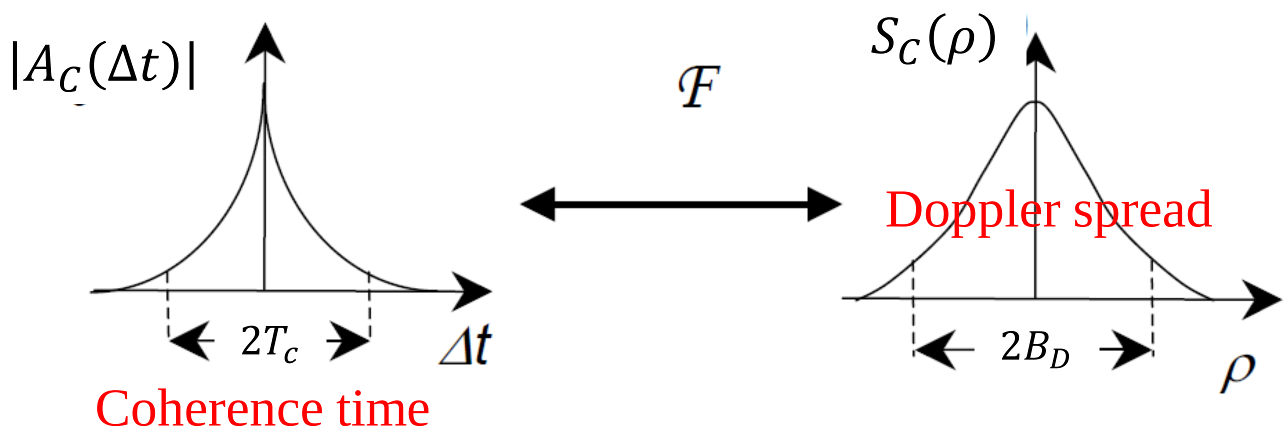
$$\bar{P}_{s0} = \int_0^\infty P_s(\gamma_s) p(\gamma_s | \bar{\gamma}_{s0}) d\gamma_s$$

- When fading environment is superposition of both fast and slow fading

  → Used in combined shadowing and flat-fading
- $\bar{P}_s$ varies slowly, locally determined by flat fading
- Declare outage when $\bar{P}_s$ above target value

Doppler Effect

- High Doppler causes channel phase to decorrelate between symbols
- Leads to an irreducible error floor for differential modulation
- Increasing power does not reduce error
- Error floor depends on $B_D T_s$



Inter-Symbol Interference

- Delay spread exceeding a symbol time causes ISI (self interference) → frequency selective fading
- ISI leads to irreducible error floor
  - Increasing signal power increases ISI power
- ISI requires that $T_s \gg \boxed{\sigma_{\tau_m}}$ $(R_S \ll \boxed{B_c})$

RMS Delay Spread

Coherence Bandwidth